ACER: A Framework on the Use of Mathematics in Upper-division Physics

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Abstract

At the University of Colorado Boulder, as part of our broader efforts to transform middle- and upper-division physics courses, we research students' difficulties with particular concepts, methods, and tools in classical mechanics, electromagnetism, and quantum mechanics. Unsurprisingly, a number of difficulties are related to students' use of mathematical tools (e.g., approximation methods). Previous work has documented a number of challenges that students must overcome to use mathematical tools fluently in introductory physics (e.g., mapping meaning onto mathematical symbols). We have developed a theoretical framework to facilitate connecting students' difficulties to challenges with specific mathematical and physical concepts. In this paper, we motivate the need for this framework and demonstrate its utility for both researchers and course instructors by applying it to frame results from interview data on students' use of Taylor approximations.

I. INTRODUCTION

Each year 6,000 physics majors graduate from US colleges and universities after having completed rigorous coursework in upper-division physics [1]. However, the PER community is accruing evidence that students throughout the major struggle with certain concepts, ideas, and tools [2–5]. These results are particularly troubling when considering the need to build on prior knowledge as our majors advance through the curriculum. Moreover, these persistent difficulties can make solving the long, complex problems in upper-division courses quite challenging.

In particular, upper-division physics students solve many problems that require sophisticated physical ideas and mathematical tools (e.g., approximation methods). Students are taught these tools in their advanced mathematics courses and solve numerous abstract mathematical exercises. Yet, students still struggle to employ these tools in their physics courses [6]. In physics, mathematical tools serve a different purpose; they are used to make inferences about physical systems. Furthermore, students must synthesize additional knowledge (e.g., conceptual physics knowledge) to apply mathematical tools to physics problems [7].

In contrast to the substantial work addressing problem solving in introductory physics courses [8], less work has been done in upper-division physics [9–11]. Prior upper-division research has focused on noting and cataloging conceptual difficulties. As students grapple with longer and more complex problems, it is increasingly important to integrate research on students' conceptual knowledge with research on their use (or misuse) of mathematical tools. Such integration will provide a more complete understanding of how students in the upper-division solve problems.

In this paper, we begin to make inroads into a synthesis of conceptual knowledge and mathematical tool usage from a theoretical perspective. Previous work in introductory physics has produced several helpful theoretical frameworks that serve a variety of purposes: coordinating multiple theories of learning [12], building on lessons from mathematics education [13], and providing a logical construction for solving problems [14]. We built upon this foundation to develop a framework to address how mathematical resources are Activated, Constructed, Executed, and Reflected upon (ACER). The ACER framework was designed to aid both instructors and researchers in exploring when and how students employ particular mathematical tools when solving canonical exercises from upper-division physics courses.

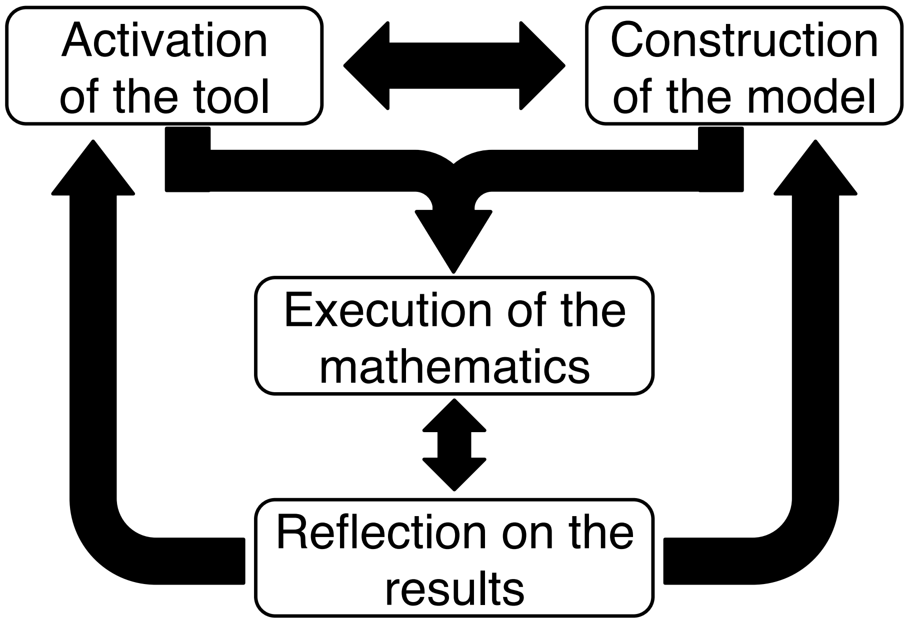


FIG. 1. A visual representation of the ACER framework.

We have also found the framework useful when developing new problems, and critiquing old ones. This paper discusses the design of this framework, demonstrates its utility with a particular example from middle-division classical mechanics (Taylor approximations), and closes with a discussion of implications and future investigations.

II. THE ACER FRAMEWORK

To help organize our observations of students' problem solving difficulties in upper-division physics courses in terms of students' conceptual knowledge and their use of mathematical tools, we have developed a theoretical framework that applies to the types of complex problems students encounter in these courses. The framework is grounded in task analysis, a method developed to uncover the tacit knowledge used by experts to perform complex tasks [15, 16], and resource theory, a model of the nature of knowledge and how it is activated and employed [17, 18]. Development of the framework was motivated by observing common difficulties in student solutions to Taylor approximation problems. We performed a task analysis on a number of these problems, which required reflecting on, documenting, and organizing the elements necessary to complete each problem. After several iterations, we organized the various elements into components that highlight the physical and mathematical concepts being activated and employed while solving Taylor approximation problems.

An astronaut is in orbit around the Earth at a distance of R from the center of the Earth. Another astronaut is in a closer orbit $(R-d)$. The difference in the strength of the gravitational potential between the astronauts is, $\Delta\phi = \frac{G\,M_E}{R} - \frac{G\,M_E}{R-d}$. Determine an approximate expression for the difference in the gravitational potential when the astronauts are very near each other.

FIG. 2. A sample problem with embedded cues that activate certain resources associated with Taylor approximation.

Our framework is organized around 4 components: *Activation of the tool*, *Construction of the model*, *Execution of the mathematics*, and *Reflection on the results* (ACER). ACER frames the challenges that students are likely to encounter in their coursework (i.e., solving "back of the book" style problems). To solve such a problem, one must determine which mathematical tool is appropriate for the model of the physical system they have constructed. Then, a series of mathematical steps are executed that facilitate the development of a solution, which must be checked for errors and compared against established or known results. A convenient way to visualize the ACER framework is shown in Fig. 1. In Fig. 1, we are not suggesting that all physics problems are solved in some clearly organized fashion, but a well articulated, complete solution involves all components of the ACER framework. The ACER framework is not general enough to be applied to experimental or open-ended problems. However, its targeted focus means that it can be operationalized for a variety of common mathematical tools (e.g., direct integration of distributed charge [19]). Below, we describe the details of each component in the context of Taylor approximations.

Activation of the tool: A problem statement contains a number of explicit or implicit cues that might activate any of a number of resources (or resource networks) associated with one or more mathematical tools [18]. Each student has their own particular association between cues and resources. Some resources that are activated might help complete the problem and others might misdirect students' efforts. For the problem shown in Fig. 2, these cues include: the goal of the problem ("determine an *approximate* expression for the difference in the gravitational potential"), as well as language and symbols that suggest some quantity (d) is much smaller than some other quantity (R). These cues are intended to activate resources associated with Taylor approximation.

Construction of the model: In physics, mathematics represents a simplified picture (i.e., a model) of a real system where each symbol has a particular physical meaning [7]. A mathematical model is typically needed to develop a solution to a physics problem. Mathematical models used in physics are typically written in a compact form (e.g., $\phi = -\int G\, dM/r$) and the identity of variables and parameters must be known or discovered. Given a specific physical situation, the use of different representations (e.g., diagrammatic or graphical) to construct elements precedes the expression of mathematical model. In some cases, a mathematical model might be provided but requires meaning be mapped onto the expression. For example, the equation given in Fig. 2 was constructed by an instructor and was provided to students, but additional work is needed to understand the model (e.g., recognizing the small expansion parameter is d/R).

Execution of the mathematics: Transforming the math structures (e.g., unevaluated integrals) in the construction component into relevant mathematical expressions (e.g., evaluated integrals) is often necessary to uncover solutions [7]. Each mathematical tool requires a specific set of steps and basic knowledge. For example, executing a Taylor approximation may require knowledge of common expansion templates (e.g., $\sin x \approx x + x^3/3! + \dots$) and how to adapt these templates to the mathematical model developed previously. Alternatively, one might need to know how to compute derivatives of complex functions. The mathematical procedures performed in this component are not, at least to experts, context free. In addition to employing base mathematical skills, experts maintain awareness of the meaning of each symbol in the expression (e.g., which symbols are constants when taking derivatives).

Reflection on the results: Expressions that are developed in upper-division physics courses are not superficial manipulations of mathematical expressions from textbooks or notes. These expressions are new entities that have predictive power and can provide greater additional insight into the behavior of the system. Reflecting on derived expressions is crucial to provide confidence in their predictions and insights (e.g., how can we know a particular expression is the correct one?). At the most basic level, reflection involves checking expressions for errors (e.g., by checking their units). Comparing the predictions to established or known results (e.g., determining its limiting behavior) is also necessary to gain confidence in these expressions. If a mistake occurred in executing the mathematics or, perhaps, some incorrect element was used in constructing the model, reflecting on the result in various ways can help uncover these errors.

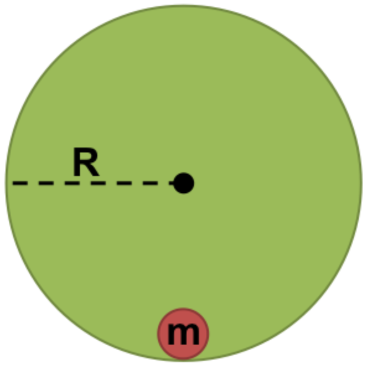
The ACER framework is most closely related to Redish's "Use of mathematics in physics" [7] and the "Logical Problem Solving Strategy" of Heller, et al. [14]; but, we distinguish it from both in its intent, its focus, and its utility. Our framework was not intended to be a model for student reasoning nor to provide a series of steps for solving problems. It provides a scaffold onto which elements of a student's solution can be organized by researchers or instructors. In doing so, ACER can help describe where students are being challenged (e.g., students produce nonsensical solutions), and can provide reasons why these difficulties exist (e.g., problems or activities focus on *Execution* while neglecting *Reflection*).

III. EMPLOYING ACER – TAYLOR SERIES

We performed eight video-taped think-aloud interviews to investigate students' use of Taylor approximations. The eight participants were physics, engineering physics, and astrophysics majors recruited from the first (6 participants) and second (2 participants) semester classical mechanics courses at CU Boulder. Taylor series had been covered in the first semester course several weeks prior to the first study. Participants tended to be the more motivated students in the course, but their exam scores reflected the full gamut of passing grades (A to D).

Both studies asked students to solve a series of Taylor approximation problems. Students' written solutions were captured using a smartpen with an embedded audio recording device (Livescribe pen). Problems included both formal math and context-laden physics questions (e.g., Fig. 3). In the first study, formal math questions were asked first, and context-laden physics questions with explicit cueing (e.g.,"perform a Taylor expansion") were asked later. We developed and, later, applied the ACER framework to organize observations from the first study. ACER demonstrated that the first study limited the possibility of observing attempts to process implicit cues. The second study began with context-laden physics questions with implicit cueing (e.g., "find an approximate expression"); formal math questions were delayed to the end. Students were asked to describe how they constructed their approximate formulae and to reflect on the physical meaning of the terms in their expressions by comparing them to known results. Video data was analyzed by identifying each key element of the framework that appeared in the students' solutions. Gestures were used to identify which parts of the solutions were being discussed in the interview. With the data we collected in these studies, we have started to organize the challenges students

A small sphere (mass, m) is free to slide inside a frictionless cylinder of radius R. If placed at the equilibrium point, $\phi = 0$ (shown below), the ball does not move.



At other non-equilibrium angles (ϕ), the gravitational potential energy for this system is given by $U(\phi) = mgR(1 - \cos\phi)$. Find an approximate expression for the gravitational potential energy for ϕ near $\phi = 0$.

FIG. 3. A context-laden Taylor approximation problem with implicit cueing used in think-aloud interviews.

face with Taylor approximations using the ACER framework.

Activation of the tool: Some upper-division physics students are just beginning to learn the "language of physics" and have not yet internalized the implicit cues that activate the use of approximation methods. No participant in the explicit cueing study failed to start a problem with a Taylor approximation. However, when solving problems with implicit cueing, 2 of 4 participants initially plugged in the given numeric value (e.g., $\phi = 0$ in Fig. 3) to determine the approximate expression (e.g., $U(\phi) \approx 0$). After these 2 participants began working the formal math problems, both asked to return to the previous context-laden problems to include approximation methods in their answers.

Construction of the model: Our studies did not deeply investigate model construction; students were often given formulae from which to work. From this limited investigation, we found that students are able to map meaning onto symbols, but they struggle to identify the relative scales of relevant variables in the problem (e.g., $d \ll R$ in Fig. 2). In both studies, participants appropriately focused their attention on variables relevant to the Taylor approximation (e.g., ϕ in Fig. 3) rather than constants (e.g., g) and parameters (e.g., R). Furthermore, no participant had significant difficulties interpreting the provided mathematical expressions.However, seven participants claimed their various expansions provided a "good" approximation to the original expressions if the variable was "small compared to 1" regardless of the expression under consideration or the presence of a natural comparative scale.

Execution of the mathematics: When computing the Taylor approximation of a function, most students elect to use the formal definition of Taylor series (Eq. 1). When requested, formulae for Taylor expansions of common functions were given to participants, but, in most instances, participants recalled or requested the formal definition. Using the formal definition was not incorrect, but led to a variety of mathematical mistakes such as taking derivatives incorrectly and forming non-polynomial expansions.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \tag{1}$$

All eight students solved the formal math problems correctly using the correct reasoning. Most students took derivatives of the equation given in Fig. 3. Only 1 student employed a common optimized method: mapping the given expression onto a certain known expansion template (e.g., $\cos(x) \approx 1 - x^2/2!$).

Reflection on the results: When prompted, most students reflect on newly constructed expressions without a clear purpose. Most were unable to connect these expressions back to the physics under investigation. In each context-laden physics problem, participants were asked to discuss their approximate expressions and reflect on its predictions. We aimed for participants to connect these expressions back to commonly understood phenomenon (e.g., projectile motion without drag). For several participants, these "forced" reflections helped uncover minor mathematical mistakes. Typically, this occurred when a participant checked the units of various terms. However, only 1 student discussed the connection between individual terms and the relevant physics (e.g., "That looks like [the potential of] a harmonic oscillator", gesturing to the approximate expression). The other seven students discussed terms superficially ("that's the drag term") or not at all ("yeah, that looks different from the usual equation").

IV. CONCLUDING REMARKS

We are using the ACER framework to help both instructors and researchers explore when and how students employ Taylor approximations when solving typical exercises from upper-division physics courses. The ACER framework helps to untangle students' sophisticated mathematical and physics difficulties and provides a convenient scaffold on which to hang these challenges. In a sense, the ACER framework addresses many important elements that define what it means to use mathematics in upper-division physics well. As such, it also provides a means to critique old and design new problems. Our investigations demonstrate that current instruction fails to enculturate students to the implicit cues that activate the use of approximation methods. Moreover, working problems with a deeper emphasis on identifying small parameters and mapping known expansion templates to mathematical models in a variety of contexts would likely benefit many students. Finally, when prompted, most students reflect on newly constructed expressions superficially, at most, checking the units of their expression. Meaningful reflections are important for connecting the math that was performed and the physics it describes. Instruction should highlight the need to gain confidence in the predictions of and insights gained from new expressions.

The ACER framework is under continual refinement. At present, where and how other important activities like coordinating representations, interpretation and prediction, and metacognition fit is an open research agenda. However, the ACER framework has already proven useful in guiding future research efforts. In future Taylor series studies, we plan to unpack the complexities of identifying small parameters (Construction) and gaining confidence in expressions (Reflection). In addition, our framework is being used in junior-level electromagnetism to explore difficulties with direct integration of charge distributions [19]. Future work will be expanded to include separation of variables in boundary value problems, and the use of direct integration and Gauss' law in gravitational problems. These studies of a variety of mathematical tools will help to further refine the ACER framework.

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