An Analytic Framework for Students’ use of Mathematics in Upper-division Physics

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Who am I?

Undergraduate: Physics and Astronomy

Graduate: Physics Education at University of Colorado
  • Defense – Spring 2015

Interests:
  • Students’ use of mathematics in physics
  • Conceptual assessment in upper-division physics
  • Upper-division course transformation
# CU Physics Education Research

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Who are we?

PER at the University of Colorado (PER@C)

Course Transformation at CU

- Transformed introductory course
  - Peer instruction (clickers) and tutorials
  - Sustained over ~10 years
Who are we?

PER at the University of Colorado (PER@C)

Course Transformation at CU

• Transformed introductory course
  • Peer instruction (clickers) and tutorials
  • Sustained over ~10 years
• Developed transformed materials for core upper-division courses
  • Research-based, tutorials, clicker questions
  • Used consistently at CU and elsewhere
  • [www.colorado.edu/sei/departments/physics.htm](http://www.colorado.edu/sei/departments/physics.htm)
Upper-division Student Difficulties
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More sophisticated physics content

Greater emphasis on mathematics

Need to connect mathematical expressions to physics concepts
Upper-division Student Difficulties

More sophisticated physics content

Greater emphasis on mathematics

Need to connect mathematical expressions to physics concepts

- “Students should be able to translate a physical description of an upper-division physics problem to a mathematical equation necessary to solve it”
- “Students should be able to achieve physical insight through the mathematics of a problem”
Use of Math in Physics

Physicists use mathematics to make inferences about physical systems.

1. map
2. process
3. interpret
4. evaluate

What do Experts do?

Model expert thinking

Heller, *et. al.* (1992)  
1. visualize the problem  
2. physics description  
3. plan the solution  
4. execute the plan  
5. check and evaluate
What do Experts do?

Model expert thinking

Heller, *et. al.* (1992)

1. visualize the problem
2. physics description
3. plan the solution
4. execute the plan
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Wright & Williams (1986)

1. What’s happening?
2. Isolate the unknown
3. Substitute
4. Evaluation

WISE Strategy
What do the Students do?

Model student thinking

**Tuminaro (2004)**

1. mathematical resources
2. epistemic games
3. frames
What do the Students do?

Model student thinking

**Tuminaro (2004)**
- (1) mathematical resources
- (2) epistemic games
- (3) frames

**Bing (2008)**
- (1) mathematical resources
- (2) epistemic frames
  - invoking authority
  - physical mapping
  - calculation
  - math consistency
Building a Framework

Guided by Experts vs. Guided by Students

• Both have virtues
• Both have drawbacks
• Can we productively leverage both?
Building a Framework

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framework – “a structure of guiding principles and assumptions about the underlying relationship between a physical concept and the mathematics necessary to describe it”

B. Wilcox, et.al. PRST-PER, 9(2) 020119 (2013)
The ACER Framework
The ACER Framework

Activation of the tool

Determine which mathematical tool is appropriate
The ACER Framework

Activation of the tool ➔ Construction of the model

Map the particular physical system onto the appropriate mathematical tool
The ACER Framework

Perform mathematical steps to reduce the solution to a form that can be interpreted.
The ACER Framework

Activation of the tool

Construction of the model

Execution of the mathematics

Reflection on the results

Interpret/check solution to gain physical insight and ensure consistency
The ACER Framework

Activation of the tool ↔ Construction of the model

Execution of the mathematics

Reflection on the results

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4 E. Redish, "WVPE" (2005); F. Reif, et al., AJP, 44 (1976); D. Wright, et al., TPT, 24 (1986); P. Heller, et al., AJP, 60 (1992)
Operationalizing ACER

Operationalization – “the process by which a particular problem or set of problems that exploit the targeted tool are mapped onto the framework”

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Modified version of Task Analysis

- Expert-driven
- Identifies important elements of a complete solution
- Refined based on student work

B. Wilcox, et.al. PRST-PER, 9(2) 020119 (2013) ; R.Catrambone, L&T Symposium (2011)
Operationalizing ACER: Integration

Multi-variable integration in electrostatics

\[ V = k \iiint \frac{dq}{|\mathbf{R}|} = k \iiint \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau' \]
Operationalizing ACER: Integration

Multi-variable integration in electrostatics

\[ V = k \iiint \frac{dq}{|\vec{R}|} = k \iiint \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \, d\tau' \]

Integration of Continuous Charge Distributions via ACER
Operationalizing ACER: Integration

Construction of the Model

\[ V = k \int \int \frac{dq}{|\vec{\mathcal{R}}|} \]

Construction

- selecting a coordinate system
- expressing \( dq \)
- expressing script-\( r \)
- identifying \( r \) and \( r' \)
- selecting limits
  - formal methods (\( r-r' \))
  - informal methods (pythagorus, law of cosines)
Operationalizing ACER: Integration

Construction of the Model

\[ V = k \iiint \frac{dq}{|\mathbf{R}|} \]

**CC1** – Use the geometry of the charge distribution to select a coordinate system
Operationalizing ACER: Integration

Construction of the Model

**CC1** – Use the geometry of the charge distribution to select a coordinate system

**CC2** – Express the differential charge element, dq, in the selected coordinates

\[ V = k \iiint dq / |\mathbf{R}| \]
Operationalizing ACER: Integration

Construction of the Model

**CC1** – Use the geometry of the charge distribution to select a coordinate system

**CC2** – Express the differential charge element, \( dq \), in the selected coordinates

**CC3** – Select integration limits consistent with the differential charge element and the physical extent of the system

\[
V = k \oint \frac{dq}{|\mathbf{R}|}
\]
Operationalizing ACER: Integration

**Construction of the Model**

**CC1** – Use the geometry of the charge distribution to select a coordinate system

**CC2** – Express the differential charge element, \( dq \), in the selected coordinates

**CC3** – Select integration limits consistent with the differential charge element and the physical extent of the system

**CC4** – Express the difference vector, \( \vec{R} \), in the selected coordinates

\[
V = k \iiint dq \frac{1}{|\vec{R}|}
\]
Findings: Integration

CC4 – expressing $|\hat{\mathcal{R}}|$.

$$V = k \iiint \frac{dq}{|\hat{\mathcal{R}}|}$$
Findings: Integration

CC4 – expressing $|\mathbf{\hat{R}}|$

$$V = k \iiint dq / |\mathbf{\hat{R}}|$$

Exams: Expressing ‘script-r’

- None: 14% (N=24)
- Incorrect: 46% (N=79)
- Correct: 40% (N=69)
Findings: Integration

CC4 – expressing $|\vec{R}|$

Emergent analysis of common difficulties:

<table>
<thead>
<tr>
<th>Difficulty</th>
<th>N</th>
<th>Percent (N=69)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ring of charge (i.e., $</td>
<td>\vec{R}</td>
<td>= \sqrt{r^2 + R^2}$)</td>
</tr>
<tr>
<td>Distance to the source or field point (i.e., $</td>
<td>\vec{R}</td>
<td>= r$ or $r'$)</td>
</tr>
<tr>
<td>No expression for $</td>
<td>\vec{R}</td>
<td>$</td>
</tr>
</tbody>
</table>

Exams: Expressing ‘script-r’

- 46% N=79
- 40% N=69
- 14% N=24

- None
- Incorrect
- Correct
~25% of students calculated V by first calculating E
~10% used Gauss’s Law to calculate E
Students typically see E first
Summary: Integration

- ~40% of students had difficulty expressing $dq$
- ~50% of students had difficulty expressing ‘script-r’
- Students struggled to coordinate math and physics resources
Summary: Integration

~10% of students explicitly made spontaneous attempts to reflect on their solutions
ACER: Taylor Series

\[ f(x) = \sum_{n} \frac{1}{n!} f^{(n)}(x_o)(x - x_o)^n \]
ACER: Taylor Series

\[ f(x) = \sum_{n} \frac{1}{n!} f^{(n)}(x_o)(x - x_o)^n \]

**Activation:** Cues imbedded in the prompt that are likely to activate resources related to Taylor series:

- **TA1** – The problem asks for a Taylor approximation directly
- **TA2** – The problem asks for an approximate expression for a complex function
- **TA3** – The problem uses language and/or symbols that imply one physical quantity is much smaller than some other physical quantity (e.g., ‘small’, ‘near’, ‘close’, or <<).
Findings: Taylor Series

TA1 – explicit prompting

“For small time, obtain an approximate expression for $x(t)$ with at least two non-zero terms using an appropriate Taylor expansion.”

Explicit Prompt

- 90% Taylor
- 10% Binomial

N=45
Findings: Taylor Series

TA1 – explicit prompting
“For small time, obtain an approximate expression for $x(t)$ with at least two non-zero terms using an appropriate Taylor expansion.”

TA2 – implicit prompting
“Find an approximate expression for the gravitational potential energy near $\varphi=0$.”

Explicit Prompt
- 90% Taylor
- 10% Binomial
- N=45

Implicit Prompt
- 50% Taylor
- 50% Other
- N=4
Summary: Taylor Series

Many students (as much as ~ 50%) had difficulty knowing when to use Taylor expansions when not explicitly prompted.
Summary: Taylor Series

>90% of students constructed correct expansions around $x_o = 0$

~50% struggled when the expansion was around $x_o \neq 0$
No students explicitly checked their solutions spontaneously.
When prompted, ~25% of students offered a correct interpretation of the physical meaning of the leading term.
ACER ...

...provides a structure to organize analysis of complex, mathematical problem solving.
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...is designed to be operationalized for specific mathematical tools in different physics contexts.
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...allowed us to coherently identify student difficulties and how these difficulties are interrelated
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...provides a structure to organize analysis of complex, mathematical problem solving.

...is designed to be operationalized for specific mathematical tools in different physics contexts.

...allowed us to coherently identify student difficulties and how these difficulties are interrelated.

...is a tool for both researchers and instructors.

    ...can facilitate the process of improving student understanding by identifying key difficulties.

...can be used as a tool to critique and design problems.
Limitations

Expert-guided

• No \textit{a priori} guarantee that ACER will span the space of student problem solving.
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Expert-guided
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Targets the intersection of math & physics
• Not purely conceptual or open-ended problems
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Targets the intersection of math & physics

• Not purely conceptual or open-ended problems

Minimal explicit incorporation of multiple representations

• Metacognition and prediction?
• Context-dependent
Thinking About Prompts

Back-of-the-book type Taylor series question

The horizontal motion of a projectile experiencing linear drag is given by,

\[ x(t) = \frac{v_x m}{b} \left( 1 - e^{-bt/m} \right). \]

i) For very small time, obtain an approximate expression for \( x(t) \) with at least two non-zero terms using an appropriate Taylor expansion.

ii) For \( t \approx \frac{m}{b} \), obtain an approximate expression for \( x(t) \) using an appropriate Taylor expansion.
Thinking About Prompts

ACER modified Taylor series question

The horizontal motion of a projectile experiencing linear drag is given by,

\[ x(t) = \frac{v_x m}{b} (1 - e^{-bt/m}). \]

i) For very small time (small compared to what?), obtain an approximate expression for \( x(t) \).

ii) According to your expression, what is the first order correction to distance the projectile would have traveled in vacuum (i.e., with no drag).
Thinking About Prompts

ACER modified Taylor series question

The horizontal motion of a projectile experiencing linear drag is given by,

\[ x(t) = \frac{v_x m}{b} (1 - e^{-bt/m}). \]

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ii) According to your expression, what is the first order correction to distance the projectile would have traveled in vacuum (i.e., with no drag).

iii) For \( t \approx m/b \), obtain an approximate expression for \( x(t) \).

iv) What is the value of your expression for \( t = m/b \)? Is this what you expect? Explain why or why not.
Thinking About Prompts

Delta functions

A. Sketch the following charge distribution:
\[ \rho(\vec{r}) = \beta \delta(z) \delta(x + 1). \]
Describe the distribution in words.
What are the units of \( \beta \)? What does \( \beta \) represent physically?
Thinking About Prompts

Delta functions

A. Sketch the following charge distribution:
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What are the units of \( \beta \)? What does \( \beta \) represent physically?

B. Using delta functions, provide a mathematical expression for the volume charge density, \( \rho(\vec{r}) \), of an infinite line of charge running parallel to the z-axis and passing through the point (1,2,0).
Be sure to define any new symbols you introduce.
THANK YOU